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On Modular Homology of Simplicial Complexes: Saturation

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Among shellable complexes a certain class is shown to have maximal modular homology, and these are the so-called *saturated* complexes. We show that certain conditions on the links of the complex imply saturation. We prove that Coxeter complexes and buildings are saturated. © 2002 Elsevier Science (USA)

Key Words: modular homology; inclusion map; simplicial complex; Cohen–Macaulay poset; shellability; Coxeter complex; building.

1. INTRODUCTION

The investigation of the modular homology of a shellable complex in general was began in [12]. There we showed that modular homology does not always behave nicely: There are shellable complexes with the same h-vector but with different modular homology. In this paper, however, we shall show that this pathological behaviour is not beyond control. We show that the homology of any shellable complex can be embedded into a well-understood module constructed purely from the shelling of the complex. This is Theorem 3.2 and the main result of Section 3.1. It follows in particular that the modular Betti numbers for an arbitrary shellable complex are bounded by functions of its h-vector only.

Shellable complexes which attain these bounds are of special interest and are called *saturated*. In this paper we investigate conditions which guarantee saturation. The main result is Theorem 4.1 which gives a sufficient

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condition under which gluing a simplex onto a saturated complex forces the resulting complex to remain saturated. Applying this condition inductively to shellable complexes suggests that the property of being saturated is *local*, that is, completely determined by the structure of the links of the complex.

Being Cohen–Macaulay, according to Reisner's theorem [14, p. 60], is also a local property. We are therefore tempted to illustrate the differences and similarities between the two local properties by the following observation: It follows from Reisner's theorem that a complex is Cohen–Macaulay if all its links are triangulations of spheres of suitable dimensions or, more generally, of manifolds whose homologies are those of bouquets of such spheres.

In contrast, it follows from Corollary 4.2 that a complex is saturated if all its links are 2-colourable triangulations of spheres of suitable dimensions, or, more generally, 2-colourable pseudomanifolds of suitable dimensions without boundary.

Here a pure complex is 2-*colourable* if its facets can be given two colours such that any two facets with a common co-dimension 1 face have different colours.

Being saturated is more restrictive than being Cohen–Macaulay, even for shellable complexes. For example, every cyclic graph is Cohen–Macaulay but only even cycles are saturated. Moreover, being saturated is not a topological invariant since it is related to 2-colourability. While the precise relationship between ordinary simplicial homology and modular simplicial homology is not yet fully understood, a partial result appears in Corollary 3.5: Shellable saturated complexes with the same modular homology have the same ordinary homology.

In Section 6 we show that finite Coxeter complexes and spherical buildings are saturated. This has lead us to investigate group actions on shellable complexes and in particular the module structure of the modular homologies of Coxeter complexes and buildings. This is the subject of a forthcoming paper. There it will be shown that if G is a finite group of Lie type with associated building Δ then the Steinberg representation of G is realized by the "top modular homology" of Δ, here see Sections 3.2 and 6.

This paper is a continuation of [12] and throughout we shall freely use the notation and results of that paper.

2. RECOLLECTING SOME PREREQUISITES

For all details of notation and assumed results we refer to [12]. The following is a short resumé only to make this paper as self-contained as possible.

Throughout F is a field of characteristic $p>0$. Let Δ be a finite complex with vertex set Ω, that is $\Delta\subseteq 2^{\Omega}$ and $x\in\Delta$, $y\subseteq x$ implies that $y\in\Delta$. The *dimension* of the face $x\in\Delta$ is $|x|-1$. If $0\leqslant k$ then M_k^{Δ} is the F-vector space with $(k-1)$-dimensional faces of Δ as basis and we put $M^{\Delta}:=\oplus_{0\leqslant k} M_k$.

The linear map $\partial: M_k^{\Delta}\to M_{k-1}^{\Delta}$ is defined by $\Delta\ni x\mapsto\sum y$ where the summation runs over all co-dimension 1 faces of x. Thus attached to Δ is the sequence

$$\mathscr{M}^{\Delta}:\quad 0\xleftarrow{\partial} M_0^{\Delta}\xleftarrow{\partial} M_1^{\Delta}\ldots\xleftarrow{\partial} M_{k-1}^{\Delta}\xleftarrow{\partial} M_k^{\Delta}\xleftarrow{\partial}\ldots\xleftarrow{\partial} M_n^{\Delta}\xleftarrow{\partial} 0.$$

For any j and $0<i<p$ we have the associated sequence

$$\ldots\xleftarrow{\partial^*} M_{j-p}^{\Delta}\xleftarrow{\partial^*} M_{j-i}^{\Delta}\xleftarrow{\partial^*} M_j^{\Delta}\xleftarrow{\partial^*} M_{j+p-i}^{\Delta}\xleftarrow{\partial^*} M_{j+p}^{\Delta}\xleftarrow{\partial^*}\ldots$$

in which ∂^* is the appropriate power of ∂. This sequence is determined by any arrow $M_l^{\Delta}\leftarrow M_r^{\Delta}$ in it and so is denoted by $\mathscr{M}_{(l,r)}^{\Delta}$. The unique arrow $M_a^{\Delta}\leftarrow M_b^{\Delta}$ for which $0\leqslant a+b<p$ is the *initial arrow* and M_b^{Δ} is the 0-*position* of $\mathscr{M}_{(l,r)}^{\Delta}$. The position of any other module in $\mathscr{M}_{(l,r)}^{\Delta}$ is counted from this 0-position and (a,b) is the *type* of $\mathscr{M}_{(l,r)}^{\Delta}$.

As $\partial^p=0$ we have $(\partial^*)^2=0$. The homology at $M_{j-i}^{\Delta}\leftarrow M_j^{\Delta}\leftarrow M_{j+p-i}^{\Delta}$ is denoted by

$$H_{j,i}^{\Delta}:=(\operatorname{Ker}\partial^i\cap M_j^{\Delta})/\partial^{p-i}(M_{j+p-i}^{\Delta})$$

and $\beta_{j,i}^{\Delta}:=\dim H_{j,i}^{\Delta}$ is the corresponding Betti number. If $\mathscr{M}_{(l,r)}^{\Delta}$ has at most one non-vanishing homology then it is *almost exact* and the only non-trivial homology then is denoted by $H_{(l,r)}^{\Delta}$. If $\mathscr{M}_{(l,r)}^{\Delta}$ is almost exact for every choice of l and r then $\mathscr{M}^{\Delta}$ is *almost p-exact*. In general, when referring to a particular sequence $\mathscr{M}_{(l,r)}^{\Delta}$, the homology at position t is denoted by H_t^{Δ} and $\beta_t^{\Delta}:=\dim H_t^{\Delta}$ is the Betti number of $\mathscr{M}_{(l,r)}^{\Delta}$ at position t.

The $(n-1)$-dimensional simplex on n vertices is denoted by Σ^n. We put $\mathscr{M}_{(l,r)}^{n}:=\mathscr{M}_{(l,r)}^{\Sigma^n}$ and as the simplex is almost p-exact, see [10], the non-trivial homology of $\mathscr{M}_{(l,r)}^{n}$ is denoted by $H_{(l,r)}^{n}$.

If Δ is any complex of dimension $n-1$ suppose that $\mathscr{M}_{(l,r)}^{\Delta}$ has type (a,b). We put

$$d_{(l,r)}^{n}:=\begin{cases}\left\lfloor\dfrac{n-a-b}{p}\right\rfloor & \text{if } n-a-b\not\equiv 0\pmod p,\\ \infty & \text{if } n-a-b\equiv 0\pmod p\end{cases}$$

and let the *weight* of $\mathscr{M}_{(l,r)}^{\Delta}$ be the integer $0<w\leqslant p$ with $w\equiv l+r-n\pmod p$. The notion of shellability and h-vectors is the usual one, the definition of k-shellable complexes may be found in Section 2 of [12].

Let M denote the F-vector space whose basis are the (finite) subsets of Ω. If $f=\sum f_x\, x\in M$ then the *support* of f is $\operatorname{supp}(f):=\bigcup\{x\colon f_x\neq 0\}$. If also $g=\sum g_y\, y\in M$ then the $\cup$-product is given by $f\cup g:=\sum_{x,y} f_x g_y\,(x\cup y)\in M$.

3. GLUING SIMPLICES: II

Here we continue our investigation into gluing simplices which was began in the section Gluing Simplices of [12]. We refer to that section as GS: I. The notation in [12] is the same as here and any detail not explained here may be found in that paper.

3.1. *Saturated Complexes*

Let Γ be an $(n-1)$-dimensional complex and let $\Delta=\Gamma\overset{k}{\cup}\Sigma^n$ be obtained by gluing Σ^n onto Γ along some k facets of Σ^n. We are considering sequences of the kind $\mathscr{M}^{*}_{(l,r)}$ where (l,r) is fixed and where $*$ is an $(n-1)$-dimensional complex. As in GS: I we put $d:=d^n_{(l,r)}$ and $u:=d^{n+k}_{(l,r)}$ where $d^n_{(l,r)}$ is the function defined at the end of the previous section. Let w denote the weight of $\mathscr{M}^{\Delta}_{(l,r)}$.

When describing the modular homology of $\Delta=\Gamma\overset{k}{\cup}\Sigma^n$ in GS: I we distinguished six cases:

G1: $1\leqslant k<w<p$, or equivalently, $d=u<\infty$;
G2: $k\equiv w\ (\mathrm{mod}\ p)$, or equivalently, $u=\infty$;
SC: $H^{\Gamma}_{u-1}=0$.

The first two are cases of 'good gluing' in Theorem 4.1 and SC is the 'special case' of Theorem 4.2 in GS: I. Here the theorems showed that *$\mathscr{M}^{\Gamma}_{(l,r)}$ and $\mathscr{M}^{\Delta}_{(l,r)}$ have the same homologies except possibly in position u where*

$$H^{\Delta}_u\simeq H^{\Gamma}_u\oplus H^{n-k}_{(l-k,\,r-k)}.$$

Note for instance that $H^{n-k}_{(l-k,\,r-k)}=0$ in the case G2.

Remaining are the three 'bad' cases of Theorem 4.2:

B1: $k\not\equiv w\equiv 0\ (\mathrm{mod}\ p)$, when $d=\infty$ and $u<\infty$,
B2: $w<k<p+w<2p$, when $d+1=u<\infty$;
B3: $k>p+w$ and $k\not\equiv w\ (\mathrm{mod}\ p)$, when $d+1<u<\infty$.

Here *$\mathscr{M}^{\Gamma}_{(l,r)}$ and $\mathscr{M}^{\Delta}_{(l,r)}$ have the same homologies except possibly in positions u and $u-1$*. To describe these we have 5-term exact gluing sequences. More specifically, for B1 and B3 the sequence is

$$\mathscr{GS}_1\colon 0\leftarrow H^{\Delta}_{u-1}\leftarrow H^{\Gamma}_{u-1}\leftarrow H^{n-k}_{(l-k,\,r-k)}\xleftarrow{\bar\theta} H^{\Delta}_u\leftarrow H^{\Gamma}_u\leftarrow 0$$

while B2 leads to

$$\mathscr{G}\mathscr{S}_2\colon 0 \leftarrow H^{\Delta}_{u-1} \leftarrow H^{\Gamma}_{u-1} \oplus H^{n}_{(l,r)} \leftarrow H^{n}_{(l,r)} \oplus H^{n-k}_{(l-k,\,r-k)} \xleftarrow{\bar{\theta}} H^{\Delta}_{u} \leftarrow H^{\Gamma}_{u} \leftarrow 0.$$

The proof of Theorems 4.1 and 4.2 showed that in either case $\bar{\theta}(H^{\Delta}_{u}) \subseteq H^{n-k}_{(l-k,\,r-k)}$. Together with exactness at H^{Δ}_{u} this means that there exists an embedding

$$H^{\Delta}_{u} \hookrightarrow H^{\Gamma}_{u} \oplus H^{n-k}_{(l-k,\,r-k)}.$$

Comparing the 'good' cases with the 'bad' ones motivates the following definition:

DEFINITION. (a) We say that the gluing $\Delta := \Gamma \overset{k}{\cup} \Sigma^n$ is *(l,r)-saturated*, or that Δ is *(l,r)-saturated* over Γ, if $H^{\Delta}_{u} \simeq H^{\Gamma}_{u} \oplus H^{n-k}_{(l-k,\,r-k)}$ in position u while $\mathscr{M}^{\Gamma}_{(l,r)}$ and $\mathscr{M}^{\Delta}_{(l,r)}$ have the same homologies in all other positions.

(b) We say that the gluing Δ has *saturated homology relative to Γ*, or that Δ is *saturated* over Γ, if $\Delta := \Gamma \overset{k}{\cup} \Sigma^n$ is (l,r)-saturated for all (l,r).

(c) The complex Δ is *(l,r)-saturated* if Δ has a shelling $\Delta_1, \Delta_2, \ldots, \Delta_m = \Delta$ in which Δ_i is (l,r)-saturated over Δ_{i-1} for every $1 < i \leqslant m$.

(d) The complex Δ has *saturated homology*, or for short is *saturated*, if Δ is (l,r)-saturated for all (l,r).

The map $\bar{\theta}\colon H^{\Delta}_{u} \to H^{n-k}_{(l-k,\,r-k)}$ in the gluing sequences will be analysed in detail later on. Important from the discussion above is a technical condition for saturation which is the basis for the main results of this paper:

LEMMA 3.1. *In the cases* B1–B3 *the gluing $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ is (l,r)-saturated if and only if $\bar{\theta}(H^{\Delta}_{u}) = H^{n-k}_{(l-k,\,r-k)}$.*

An important general consequence of the Theorems 4.1 and 4.2 in GS: I, together with the analysis above, follows by induction. The next theorem and its corollary may in fact serve as an alternative definition of saturation:

THEOREM 3.2. *Let Δ be an $(n-1)$-dimensional shellable complex with h-vector $(h_0,\ldots,h_n)$. For fixed (l,r) let H^{Δ}_{t} denote the homology of $\mathscr{M}^{\Delta}_{(l,r)}$ at position t, put $d := \min\{\, d^{n}_{(l,r)}, d^{n+1}_{(l,r)} \,\}$ and let w be the weight of $\mathscr{M}^{\Delta}_{(l,r)}$. Then $H^{\Delta}_{t} = 0$ for $t < d$ and for all $s \geqslant 0$ there is an embedding*

$$(*)\colon\quad H^{\Delta}_{d+s} \hookrightarrow \bigoplus_{j=w+(s-1)\,p+1}^{w+sp} \left[H^{n-j}_{(l-j,\,r-j)}\right]^{h_j}.$$

Furthermore, Δ is (l, r)-saturated if and only if $(*)$ is an isomorphism for all $s \geqslant 0$. In particular, Δ is saturated if and only if $(*)$ is an isomorphism for all $s \geqslant 0$ and all (l, r).

COROLLARY 3.3. *Let Δ' and Δ be shellable complexes of the same dimension and with the same h-vector. Suppose that Δ is (l, r)-saturated. Then the Betti numbers of $\mathscr{M}^{\Delta'}_{(l,r)}$ and $\mathscr{M}^{\Delta}_{(l,r)}$ satisfy*

$$(*):\quad \beta^{\Delta'_t} \leqslant \beta^{\Delta}_t$$

for all $t \in \mathbf{Z}$. Furthermore, Δ' is (l, r)-saturated if and only if $(*)$ is an equality for each $t \in \mathbf{Z}$. In particular, if Δ is saturated then Δ' is saturated if and only if $(*)$ is an equality for all (l, r) and all $t \in \mathbf{Z}$.

Remark 1. In Theorem 3.2 we use the convention that $[H]^0$ is the zero module.

2. There are examples of complexes which are (l, r)-saturated for certain values of (l, r) but not for others. For instance, when $p = 3$ and Δ is the cone over a cyclic graph with 7 vertices then Δ is $(1, 3)$-saturated but not $(1, 2)$-saturated.

3. Saturation is defined with respect to a prime p and it is not clear if there are complexes which are saturated for some primes but not for others. See also [7, 5.1.25, p. 214].

4. For shellable Δ it follows from results of [12] that $H^{\Delta}_t = 0$ for all $t < d$. Moreover:

- All 1-shellable complexes are saturated, see Corollary 5.3 in [12].
- If Δ is k-shellable and if $\mathscr{M}^{\Delta}_{(l,r)}$ is a sequence of weight at least k then Δ is (l, r)-saturated and almost exact, see Theorem 5.1 in [12]. For instance, when $p > 2$ and Δ is the cone over a cyclic graph then Δ is 2-shellable and so the only parameters for which Δ is possibly not (l, r)-saturated come from sequences of weight 1, just as in the example above where the cycle has odd length. We shall see later that cones over even cycles are always saturated.

3.2. *Examples and Further Observations*

We conclude with several comments which may illustrate saturation. As we have just seen, for a saturated complex all Betti numbers are determined entirely by the shelling vector. For instance, if Δ is a 5-dimensional complex with h-vector $(h_0, h_1, \ldots, h_6)$ which is saturated for $p = 3$ then its Betti numbers are as shown in Table I.

TABLE I

The 3-Modular Betti Numbers of a Saturated 5-Complex

(l, r)	w			
(1,2)	3	$\beta_{4,2}=h_1+h_2$;		$\beta_{5,1}=h_4+h_5$
(1,3)	1	$\beta_{3,2}=h_0$;	$\beta_{4,1}=h_2+h_3$;	$\beta_{6,2}=h_5+h_6$
(2,3)	2	$\beta_{3,1}=h_0+h_1$;	$\beta_{5,2}=h_3+h_4$;	$\beta_{6,1}=h_6$

If Δ, as before with h-vector $(h_0, h_1, \ldots, h_6)$, is saturated for $p=5$ then the Betti numbers are as shown in Table II below.

We have no general existence theorem or construction which produces, for given complex with known h-vector and given prime p, any complex with the same h-vector which is saturated for p. In particular, there may be no saturated complex with that given h-vector at all. Such questions may be interesting to investigate.

One further and significant observation is that $\beta_{6,1}$ is the same in both tables, and equal to h_6. In fact, for any $(n-1)$-dimensional saturated complex it is seen easily that the modular Betti number $\beta_{n,1}$ is the last component of the shelling vector.

This is a surprising parallel to the situation of the ordinary homology which is defined in relation to the usual boundary operator of simplicial geometry. Here a routine application of the Mayer–Vietoris sequence shows that shellable complexes are Cohen–Macaulay. This means that the complex has a single non-trivial simplicial homology located at the top. The dimension of this homology, sometimes called the Steinberg module of the complex, is also the last component of the shelling vector. We hope to

TABLE II

The 5-Modular Betti Numbers of a Saturated 5-Complex

(l, r)	w		
(1,2)	2	$\beta_{2,1}=8h_0+3h_1$;	$\beta_{6,4}=h_3+h_4+h_5+h_6$
(1,3)	3	$\beta_{3,2}=13h_0+8h_1+3h_2$;	$\beta_{6,3}=h_4+h_5+h_6$
(1,4)	4	$\beta_{4,3}=8h_0+8h_1+5h_2+2h_3$;	$\beta_{6,2}=h_5+h_6$
(1,5)	5	$\beta_{5,4}=3h_1+3h_2+2h_3+h_4$;	$\beta_{6,1}=h_6$
(2,3)	4	$\beta_{3,1}=5h_0+5h_1+3h_2+h_3$;	
(2,4)	5	$\beta_{4,2}=5h_1+5h_2+3h_3+h_4$;	
(2,5)	1	$\beta_{2,2}=8h_0$;	$\beta_{5,3}=3h_2+3h_3+2h_4+h_5$
(3,4)	1	$\beta_{3,4}=5h_0$;	$\beta_{4,1}=2h_2+2h_3+h_4$
(3,5)	2	$\beta_{3,3}=13h_0+5h_1$;	$\beta_{5,2}=2h_3+2h_4+h_5$
(4,5)	3	$\beta_{4,4}=8h_0+5h_1+2h_2$;	$\beta_{5,1}=h_4+h_5$

explain this phenomenon and establish conditions which guarantee that the Steinberg module is indeed isomorphic to the top modular homology. This is known to be true in the case of buildings, and extensions of this result will be the subject of a forthcoming paper.

As Corollary 3.3 shows, the saturation of a complex can be decided from its Betti numbers. However, it is worthwhile to mention a related observation which we illustrate first by an example.

Let Δ be a shellable 5-dimensional simplicial complex and let

$$\mathcal{M}^{\Delta}_{(0,2)}: \quad 0 \leftarrow M^{\Delta}_0 \leftarrow M^{\Delta}_2 \leftarrow M^{\Delta}_3 \leftarrow M^{\Delta}_5 \leftarrow M^{\Delta}_6 \leftarrow 0$$

be its 3-modular sequence of type (0, 2). The Euler characteristic of $\mathcal{M}^{\Delta}_{(0,2)}$ is $\chi^{\Delta}_{(0,2)} := f_0 - f_2 + f_3 - f_5 + f_6$ and using the well-known relation [3, 14]

$$(*): \quad f_j = \sum_{k=0}^{n} \binom{n-k}{j-k} h_k$$

we can write this characteristic in terms of the h-vector of Δ as

$$\chi^{\Delta}_{(0,2)} = (h_0 + h_1) - (h_3 + h_4) + h_6.$$

By the trace formula $\chi^{\Delta}_{(0,2)}$ is also the alternating sum of Betti numbers, and so one may make a naive conjecture that $b_2 := h_0 + h_1$, $b_1 := h_3 + h_4$ and $b_0 := h_6$ could be the non-zero Betti numbers of $\mathcal{M}^{\Delta}_{(0,2)}$. The preceding table shows that this conjecture, though evidently false in general, does hold for *saturated* 5-dimensional complexes. In fact, using the expression for $\dim H^n_{(l,r)}$ from Theorem 2.1 in [12] it is easy to show that this observation holds true for all saturated complexes and every p.

Moreover, a shellable complex is saturated if and only if this procedure works. To make this statement precise let Δ be shellable and consider the following algorithm for $\mathcal{M}^{\Delta}_{(l,r)}$:

1. Using the relation $(*)$ express its Euler characteristic $\chi^{\Delta}_{(l,r)}$ in terms of the h_i.

2. Arrange the h_i in order of ascending indices. Starting from the right, compute the sum $\pm b_0$, with $b_0 > 0$, formed by the first group of consecutive terms carrying the same sign. Continue with the next groups, thus defining $b_1 > 0$, $b_2 > 0$,... and so on, resulting in $\pm\, \chi^{\Delta}_{(l,r)} = \ldots + b_2 - b_1 + b_0$.

In this procedure one observes that consecutive h_i terms carry the same sign with groups always separated by a single missing index, and that each group, apart possibly from the first and the last, consists of $p-1$ terms, just as in the example. It is also known that if h_{i_0} and h_{i_1} denote the first, respectively last, non-zero entries of the h-vector of a shellable complex then all intermediate components are positive, that is $h_i > 0$ if $i_0 \leqslant i \leqslant i_1$, see Theorem 5.1.15 in [7].

The following result is a straightforward consequence of Corollary 6.3 and Theorem 6.2, and its proof is left to the reader. It may serve as an alternative definition of saturation: On the one hand it gives a ready formula for the Betti numbers of a saturated complex with known h- or f-vector. On the other hand, if the Betti numbers and the f- or h-vector are known, then it provides an exact condition for saturation:

COROLLARY 3.4. *Let Δ be a shellable complex. For given (l, r) consider the sequence $\mathscr{M}^{\Delta}_{(l,r)}$ with Betti numbers ..., β_{t-2}, β_{t-1}, β_t and define ..., b_2, b_1, b_0 as above. Then Δ is (l, r)-saturated if and only if ..., $\beta_{t-2} = b_2$, $\beta_{t-1} = b_1$, $\beta_t = b_0$.*

This observation can be taken further. Using $(*)$ the h-vector can be defined even for a non-shellable Δ, see [14, p. 58]. For given f-vector one may therefore define *formal Betti numbers* $b^{\Delta}_{j,i}$ for any prime p. To do this consider the sequence $\mathscr{M}^{\Delta}_{(j-i,j)}$, compute ..., b_2, b_1, b_0 according to the algorithm above and select for $b^{\Delta}_{j,i}$ the corresponding term among the b's. Alternatively, work out the dimension

$$\sum_{j=w+(s-1)p+1}^{w+sp} h_j \beta^{n-j}_{(l-j,\, r-j)}$$

of H^{Δ}_{d+s} in Theorem 3.2. The corollary now suggests a more general notion of saturation for pure complexes: Namely that Δ is (l, r)-saturated for p if and only if the actual Betti numbers of $\mathscr{M}^{\Delta}_{(l,r)}$ are equal to the corresponding formal Betti numbers. As before Δ then is saturated for p if and only if it is (l, r)-saturated for all (l, r).

This extension of saturation to non-shellable complexes may turn out to be interesting. For $p = 3$ examples of such saturated complexes include well-known non-shellable triangulations of 3-balls, such as the *'knotted hole ball'* described by Furch in 1924, and the '2-*roomed house*' constructed by Bing in 1964. Both are Cohen–Macaulay with $f = (1, 380, 1929, 2722, 1172)$ and $f = (1, 480, 2511, 3586, 1554)$ respectively, see [8, 9, 15].

Other interesting examples are the non-shellable triangulation of the projective plane (with $f = (1, 6, 15, 10)$, not Cohen–Macaulay) and of the dunce hat (with $f = (1, 8, 24, 17)$, Cohen–Macaulay). These are both $(1, 2)$- and $(2, 3)$-saturated for $p = 3$ but not $(1, 3)$-saturated. In contrast, the 16-vertex triangulation of the Poincaré 3-sphere due to Björner and Lutz (non-shellable, Cohen–Macaulay, see [8]) is not (l, r)-saturated for $p = 3$ and any (l, r). Note that the upper bounds for Betti numbers in Corollary 3.3 hold true in each of these three complexes.

We return to shellable complexes and the relationship between the ordinary homology and the modular homology. Suppose that Δ is shellable and saturated with respect to some prime p, and that all modular Betti numbers $\beta^{\Delta}_{j,i}$ are known. Thus $b^{\Delta}_{j,i} = \beta^{\Delta}_{j,i}$ for all j, i and we may invert the relation between the $b^{\Delta}_{j,i}$ and the h-vector to compute the h- and hence the f-vector of Δ. Therefore the ordinary Euler characteristic $\pm\chi(\Delta) := f_0 - f_1 + f_2 - \ldots$ is known.

Let $\bar{\beta}^{\Delta}_i$ denote the ordinary Betti numbers of Δ, defined in relation to the usual boundary operator of simplicial geometry. As Δ is shellable it is Cohen–Macaulay and hence $\bar{\beta}^{\Delta}_0 = \ldots = \bar{\beta}^{\Delta}_{n-1} = 0$ while $\bar{\beta}^{\Delta}_n = \pm\chi(\Delta)$ if $(n-1)$ denotes the dimension of Δ. Hence we have the following:

Corollary 3.5. *Let Δ and Δ^* be (shellable) complexes of the same dimension which are saturated for some prime p. If Δ and Δ^* have the same modular Betti numbers then Δ and Δ^* have the same f- and h-vectors and have the same ordinary Betti numbers.*

4. A CONDITION FOR SATURATION: THE MAIN THEOREM

As we have seen, saturation is automatic in any of the good cases and in the bad cases it is equivalent to the conditions $\bar{\theta}(H^{\Delta}_u) = H^{n-k}_{(l-k,\, r-k)}$ in the gluing sequence. To investigate this condition further we need some additional notation.

Let σ denote the vertex set of Σ^n and let $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$. Then the *restriction* $\operatorname{res}(\sigma)$ is the set of all vertices $\beta \in \sigma$ such that $\sigma \setminus \{\beta\}$ is contained in Γ, see Björner [4]. So $\operatorname{res}(\sigma)$ is a $(k-1)$-face of Σ^n and one may regard it as the 'outer face' under gluing. Its complement $t(\sigma) := \sigma \setminus \operatorname{res}(\sigma)$ is the 'inner face' under gluing. If x is a face of Δ then the subcomplex $\operatorname{star}_{\Delta}(x)$ is generated by all facets that contain x and $\operatorname{link}_{\Delta}(x)$ is the subcomplex of all faces of $\operatorname{star}_{\Delta}(x)$ that do not contain x. So the dimension of $\operatorname{link}_{\Delta}(x)$ is $n - |x| - 1$. The situation is best illustrated by Fig. 1.

Here A is the intersection complex $\Gamma \cap [\sigma]$ which we shall need later. The restriction is $\operatorname{res}(\sigma) = \{\beta_1, \beta_2\}$ and the inner face is $t(\sigma) = \{\beta\}$. It is useful to view arrows as embeddings and so we regard A, Σ^n and Γ as subcomplexes of Δ. Also, $\operatorname{link}_{\Delta}(t(\sigma))$ is the boundary $(\delta_1, \delta_2, \delta_3, \delta_4)$.

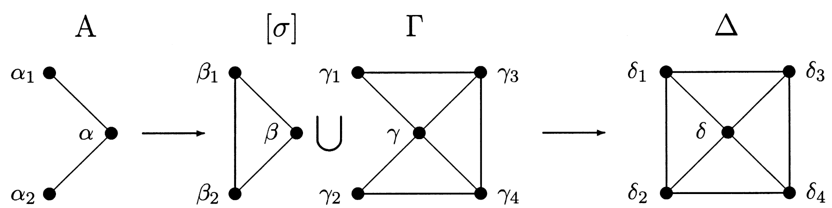

FIG. 1 Gluing Σ^3 onto Γ

The main theorem of this section now follows. When saying that $\operatorname{res}(\sigma)$ is a 1-*cycle of* Δ *relative to* $\operatorname{link}_\Gamma(t(\sigma))$ we mean that there is some $f \in M_k^\Gamma \subset M^\Delta$ such that $\operatorname{supp}(f) \cap t(\sigma) = \varnothing$, $f \cup t(\sigma) \in M^\Gamma$ and $\partial(\operatorname{res}(\sigma) + f) = 0$.

THEOREM 4.1. *Let Γ be a complex and let $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$. Suppose that* $\operatorname{res}(\sigma)$ *is a 1-cycle of Δ relative to* $\operatorname{link}_\Gamma(t(\sigma))$. *Then Δ is saturated relative to Γ.*

The proof is quite technical and is relegated to the next section.

DEFINITION. Let Δ be a pure $(n-1)$-dimensional complex with facets $\sigma_1, \ldots, \sigma_m$. Then Δ is *null over F with respect to ∂*, or just *null* for short, if there are non-zero $c_1, \ldots, c_m \in F$ such that $\partial(c_1\sigma_1 + \ldots + c_m\sigma_m) = 0$.

Note, nullness with respect to ∂ is in no obvious relationship to nullness with respect to the ordinary boundary map of simplicial geometry. However, occasionally complexes are null in both senses. We say that a complex is 2-*colourable* if its facets can be 2-coloured in such a way that facets with a common co-dimension 1 face have different colours. Further, in a *pseudomanifold without boundary*, see Definition 3.15 in [14], each co-dimension 1 face is contained in exactly 2 facets. Therefore a 2-colourable pseudomanifold without boundary is null with respect to the boundary map and with respect to ∂: Choose all $c_i = 1$ for the first case and $c_i = \pm 1$, suitably according to the 2-colouring, in the second case. In particular, even cyclic graphs are null over every field, and odd cyclic graphs are null only over fields of characteristic 2.

COROLLARY 4.2. *Let Γ be a complex and let $\Delta = \Gamma \overset{k}{\cup} [\sigma]$ for some $k \geqslant 1$. Suppose that* $\operatorname{link}_\Delta(t(\sigma))$ *is null. (In particular, suppose that* $\operatorname{link}_\Delta(t(\sigma))$ *is a 2-colourable triangulation of a sphere, or a 2-colourable pseudomanifold without boundary.) Then Δ is saturated relative to Γ.*

Proof. It follows from the definition of links that $\operatorname{res}(\sigma) \in \operatorname{link}_\Delta(t)$. So let $\sigma_1 := \operatorname{res}(\sigma), \ldots, \sigma_m$ be the facets of $\operatorname{link}_\Delta(t)$ and let $c_1, \ldots, c_m \in F$ all be non-zero such that $\partial(c_1\sigma_1 + \ldots + c_m\sigma_m) = 0$. Now put $f := c_1^{-1}[c_2\sigma_2 + \ldots + c_m\sigma_m]$. The result follows from Theorem 4.1. ▮

EXAMPLE. The $(n-1)$-dimensional *hyperoctahedron* or *cross-polytope* has vertex set $\Omega := \{\alpha_i, \beta_i : 1 \leqslant i \leqslant n\}$ and faces formed by all Ω-subsets which contain at most one of α_i, β_i for each $1 \leqslant i \leqslant n$. (Thus it is obtained by performing successive suspensions over vertex pairs α_i, β_i, or alternatively, as the dual of the $(n-1)$-dimensional cube.) It is easy to see that this complex is shellable and that all links in a shelling are null. So the hyperoctahedron is saturated for all primes.

5. THE PROOF OF THEOREM 4.1

On first reading this section may be omitted, the proof of Theorem 4.1 is its only purpose. From the discussion in Section 3.1 and Lemma 3.1 it is clear that it suffices to prove the following more technical version of Theorem 4.1:

THEOREM 5.1. *If $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ fix some (l, r) in one of the cases* B1–B3. *If $r^\sigma := \operatorname{res}(\sigma)$ is the restriction and $t := \sigma \backslash r^\sigma$ its complement suppose there exists some $f \in M_k^\Gamma \subset M^\Delta$ with* $\operatorname{supp} f \cap t = \emptyset$, $f \cup t \in M^\Gamma$ *and $\partial(r^\sigma + f) = 0$. Then $\bar{\theta}(H_u^\Delta) = H_{(l-k,\, r-k)}^{n-k}$.*

The proof is arranged in three parts: First, we analyse the map $\bar{\theta}$. Then we examine the module $H_{(l-k,\, r-k)}^{n-k}$ in more detail to find its generators, and finally we show that $\bar{\theta}(H_u^\Delta) = H_{(l-k,\, r-k)}^{n-k}$. We shall see that it is irrelevant for the proof which of the bad cases occurs.

1. *The definition of the connecting map $\bar{\theta}$.* For $\Delta = \Gamma \overset{k}{\cup} [\sigma]$ let $\mathrm{A} := \Gamma \cap [\sigma]$ be a part of the boundary of $[\sigma]$ consisting of k faces of dimension $(n-2)$, see again Fig. 1. Associated to $\Gamma \overset{k}{\cup} [\sigma]$ is the Mayer–Vietoris sequence

$$0 \leftarrow \mathscr{D} \overset{\psi}{\leftarrow} \mathscr{C} \oplus \mathscr{B} \overset{\phi}{\leftarrow} \mathscr{A} \leftarrow 0$$

where $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ denote $\mathscr{M}_{(l,r)}^{\mathrm{A}}$, $\mathscr{M}_{(l,r)}^{\Sigma^n}$, $\mathscr{M}_{(l,r)}^{\Gamma}$ and $\mathscr{M}_{(l,r)}^{\Delta}$ respectively. Written as a diagramme this sequence is shown in Fig. 2, where ∂^*, as before, stands for whatever power of ∂ is needed in the context. There are the natural embeddings shown in Fig. 3, and for $a \in \mathscr{A}$ we indicate its images in $\mathscr{B}$ and $\mathscr{C}$ by a_B and a_C respectively. The same convention applies to $b \in \mathscr{B}$ and $c \in \mathscr{C}$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots & A_{u-1} \arrow[l] \arrow[d, "\phi_{u-1}"] & A_u \arrow[l, "\partial^*"'] \arrow[d, "\phi_u"] & A_{u+1} \arrow[l, "\partial^*"'] \arrow[d, "\phi_{u+1}"] & \cdots \arrow[l] \\
\cdots & B_{u-1} \oplus C_{u-1} \arrow[l] \arrow[d, "\psi_{u-1}"] & B_u \oplus C_u \arrow[l, "\partial^*"'] \arrow[d, "\psi_u"] & B_{u+1} \oplus C_{u+1} \arrow[l, "\partial^*"'] \arrow[d, "\psi_{u+1}"] & \cdots \arrow[l] \\
\cdots & D_{u-1} \arrow[l] \arrow[d] & D_u \arrow[l, "\partial^*"'] \arrow[d] & D_{u+1} \arrow[l, "\partial^*"'] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

FIG. 2. The Mayer–Vietoris Sequence.

$$
\begin{tikzcd}
 & \mathcal{A} \arrow[dl] \arrow[dr] & \\
\mathcal{B} \arrow[dr] & & \mathcal{C} \arrow[dl] \\
 & \mathcal{D} &
\end{tikzcd}
$$

FIG. 3. The Natural Embeddings.

The homomorphisms ϕ and ψ are now given by $\phi(a) := (a_B, -a_C)$ and $\psi(b, c) := b_D + c_D$, see also [13, p. 143].

Now the gluing sequence is just an interval of the long homological sequence

$$\cdots \leftarrow H^{\Delta}_{u-1} \leftarrow H^{\Gamma}_{u-1} \oplus H^{n}_{u-1} \leftarrow H^{\mathrm{A}}_{u-1} \xleftarrow{\bar{\theta}} H^{\Delta}_{u} \leftarrow H^{\Gamma}_{u} \oplus H^{n}_{u} \leftarrow \cdots$$

associated with Fig. 2 and $\bar{\theta}$ is the usual connecting map. Its definition is standard and may be found in any textbook of homological algebra or algebraic topology, see for example [13]. We include it only for the sake of completeness.

The map $\bar{\theta}$: $H^{\Delta}_{u} \to H^{\mathrm{A}}_{u-1}$ is induced naturally by the map $\theta: D_u \to A_{u-1}$ which can be defined as follows: Let $[d] \in H^{\Delta}_{u}$ be a class of homologies with representative $d \in D_u$, thus $\partial^*(d) = 0$. To define $\theta(d)$ note that since ψ: $B_u \oplus C_u \to D_u$ is a surjection, there is $(b, c) \in B_u \oplus C_u$ such that $\psi(b, c) = d$ (evidently, (b, c) is not necessarily unique). So let $(b', c') \in B_{u-1} \oplus C_{u-1}$ be given by $(b', c') = \partial^*(b, c) = (\partial^*(b), \partial^*(c))$. It follows from the commutativity of the diagram that (b', c') is in the image of the monomorphism ϕ: $A_{u-1} \to B_{u-1} \oplus C_{u-1}$. So there is a unique $a \in A_{u-1}$ such that $\phi(a) = (b', c')$. Now define θ via $\theta(d) = a$. Again from the commutativity of the diagram it follows that $\partial^*(a) = 0$. So let $\bar{\theta}[h] = [a] \in H^{\mathrm{A}}_{u-1}$. It is a trivial matter to show by 'diagram chasing' that the last definition is independent of the choices in the definition of θ.

For short one can say that $\bar{\theta}$ is *induced by* $\theta = \phi^{-1}\partial^*\psi^{-1}$ where $\psi^{-1}(d)$ is *any* pre-image of d.

2. *The structure of* H^{A}_{u-1}. We recall a decomposition of H^{A}_{u-1} from Theorem 3.1 of [12]. First, note that the natural inclusion $\mathrm{A} \subseteq \mathrm{B}$ implies the exact sequence

$$(**): \quad 0 \leftarrow \mathscr{B}/\mathscr{A} \xleftarrow{\vartheta} \mathscr{B} \xleftarrow{\iota} \mathscr{A} \leftarrow 0$$

where the terms in $\mathscr{B}/\mathscr{A}$ are quotients of the form $M^{n}_{j}/M^{\mathrm{A}}_{j}$. One notes easily that $M^{n}_{j}/M^{\mathrm{A}}_{j} \simeq M^{n-k}_{j-k}$ and so $\mathscr{B}/\mathscr{A}$ is a sequence with parameters

$(l-k, r-k)$. Moreover, the surjection ϑ: $M_j^n \to M_{j-k}^{n-k}$ is defined as follows, see also the proof of Lemma 3.2 in [12]: Let r^σ be the restriction associated with $\Delta = \Gamma \overset{k}{\cup} \Sigma^n$ (so that $|r^\sigma| = k$). Evidently, every element $x \in M_j^n$ can be written uniquely as $x = r^\sigma \cup y + z$ where $y \in M_{j-k}^{n-k}$. Define ϑ by $\vartheta(x) := y$.

To find H_{u-1}^{A} it remains to note that $\mathscr{B}/\mathscr{A}$ is almost exact with non-trivial homology $H_{(l-k, r-k)}^{n-k}$. Therefore, in the bad cases the long homology sequence arising from (**) is

$$0 \leftarrow H_{u-1}^n \leftarrow H_{u-1}^{\mathrm{A}} \overset{\bar{\vartheta}}{\leftarrow} H_{(l-k, r-k)}^{n-k} \leftarrow 0$$

so that $H_{u-1}^{\mathrm{A}} \simeq H_{u-1}^n \oplus H_{(l-k, r-k)}^{n-k}$. The standard construction for a connecting map may be applied to the inclusion $\bar{\vartheta}$ and repeating the arguments above, we obtain the following which will be crucial for the last part of our proof:

Lemma 5.2. *In the decomposition* $H_{u-1}^{\mathrm{A}} \simeq H_{u-1}^n \oplus H_{(l-k, r-k)}^{n-k}$ *the component* $H_{(l-k, r-k)}^{n-k}$ *is spanned by elements of the form* $[\partial^*(r^\sigma \cup e)] \in H_{u-1}^{\mathrm{A}}$ *where* $e \in M_{j-k}^{n-k}$ *and* $\partial^*(e) = 0$.

3. *Completing the proof.* Let $r^\sigma \subseteq \sigma$ be as before and put $t := \sigma \setminus r^\sigma$. We identify r^σ with an element of $M^n := M^{\mathrm{B}}$ and the inclusion $\mathrm{B} \subseteq \Delta$ induces the identification $M^B \ni b \mapsto b_D \in M^\Delta$.

Let $\theta = \phi^{-1}\partial^*\psi^{-1}$ be defined as before. It follows from Lemma 5.2 that it is enough to prove that for every $e \in M_{j-k}^{n-k}$ with $\mathrm{supp}(e) \subseteq t$ and $\partial^*(e) = 0$ one can find some $[h] \in H_u^\Delta$ such that $\theta(h) = \partial^*(r^\sigma \cup e)$.

For this let $f \in M_k^\Gamma \subset M^\Delta$ be as in Theorem 5.1. Now take $h := (r^\sigma + f)_D \cup e_D \in M^\Delta$. Since $\partial^*(e) = 0$, and e_D and $(r^\sigma + f)_D$ have non-intersecting supports, also $\partial^*(h) = 0$ and so the corresponding class $[h]$ is in H_u^Δ.

From Theorem 2.2 in [10] we derive the fundamental fact that any e with $\partial^*(e) = 0$ can be written as a linear combination of elements of the form $s \cup v$ where s is a set, $\partial^* s = 0$ and $\partial v = 0$. This means that we can suppose that $e = s \cup v$ and so

$$\begin{aligned}
\theta(h) &= \phi^{-1}\partial^*\psi^{-1}\,[(r^\sigma + f) \cup e]_D \\
&= \phi^{-1}\partial^*([r^\sigma \cup s \cup v]_B, [f \cup s \cup v]_C) \\
&= \phi^{-1}(\partial^*(r^\sigma \cup s) \cup v_B, \partial^*(f \cup s) \cup v_C) \\
&= \phi^{-1}(\partial^*(r^\sigma \cup s) \cup v_B, -\partial^*(r^\sigma \cup s) \cup v_C) \\
&= \partial^*(r^\sigma \cup s) \cup v_A \\
&= \partial^*(r^\sigma \cup s \cup v) \\
&= \partial^*(r^\sigma \cup e).
\end{aligned}$$

The equality $\partial^*(f \cup s) = -\partial^*(r^\sigma \cup s)$ follows from the fact that $\partial^* s = 0$ and so $\partial^*((r^\sigma + f) \cup s) = (r^\sigma + f) \cup \partial^* s = 0$. This completes the proof of Theorems 5.1 and 4.1. ∎

6. COXETER COMPLEXES AND BUILDINGS

In this section we show that Coxeter complexes and buildings are saturated in any characteristic. Without going into further details we shall assume the following facts:

- Buildings and Coxeter complexes are shellable [4];
- Finite Coxeter complexes are triangulations of spheres [6, p. 62];
- Every link of a Coxeter complex or a building is again a Coxeter complex or, respectively, a building [6, pp. 60,79];
- The ordinary Euler characteristic of an $(n-1)$-dimensional shellable complex Δ (with regards to the simplicial boundary map, thus the alternating sum of the face numbers) is equal to $h_n(\Delta)$, up to sign.

In addition, for a Coxeter complex we have the following:

LEMMA 6.1. *Coxeter complexes are 2-colourable and so are null with respect to ∂ for any field of characteristic $p > 0$.*

Proof. If Δ is a Coxeter complex let x_0 be the facet corresponding to the identity in its reflection group. It is well-known that one may use the involutory generators of this group to define a distance function d on the facets of Δ such that $d(x_0, x) - d(x_0, y) = \pm 1$ if and only if x and y meet in a co-dimension 1 face. For $i = 0, 1$ now put $C_i := \{ y : d(x_0, y) \equiv i \bmod 2 \}$. Then C_0 and C_1 are the classes of a 2-colouring of Δ. Since Δ is a triangulation of a sphere it has no boundary and hence is null according to the remarks following the definition of nullness earlier. ∎

Let now Δ be a Coxeter complex with shelling $\Delta_1 = [\sigma_1] \approx \Sigma^n, \Delta_2, \ldots, \Delta_m = \Delta$ where $\Delta_i = \Delta_{i-1} \overset{k_i}{\cup} [\sigma_i]$ for $0 < i \leqslant m$ and with $[\sigma_i] \approx \Sigma^n$.

LEMMA 6.2. *Let* $\operatorname{res}(\sigma_i)$ *be the restriction and let* $t_i = \sigma_i \setminus \operatorname{res}(\sigma_i)$ *be its complement. Then* $\operatorname{link}_{\Delta_i}(t_i) = \operatorname{link}_{\Delta}(t_i)$.

Proof. Let $\Delta_i = \Delta_{i-1} \cup \sigma_i$ be a k-gluing. As Δ_i is shellable also $\operatorname{link}_{\Delta_i}(t_i)$ is shellable and hence Cohen–Macaulay. Therefore $\operatorname{link}_{\Delta_i}(t_i)$ is homotopic to a bouquet of m spheres, all of dimension $k-1$. In the shelling of $\operatorname{link}_{\Delta_i}(t_i)$ induced by the shelling of Δ, note that the face $\operatorname{res}(\sigma_i)$ is a homology face (that is, it is its own restriction) and so $h_k(\operatorname{link}_{\Delta_i}(t_i)) \geqslant 1$. As

$h_k(\mathrm{link}_{\Delta_i}(t_i))$ is the Euler characteristic of $\mathrm{link}_{\Delta_i}(t_i)$, up to sign, it follows that $m \geqslant 1$.

On the other hand, $\mathrm{link}_{\Delta_i}(t_i)$ is a subcomplex of the $(k-1)$-dimensional complex $\mathrm{link}_{\Delta}(t_i)$. The latter is again a Coxeter complex, by the general assumption from the beginning of this section, and hence a triangulation of a sphere. But then $m \leqslant 1$ and so $\mathrm{link}_{\Delta_i}(t_i)$ is homotopic to a sphere. Since $\mathrm{link}_{\Delta}(t_i)$ triangulates the sphere, every facet in that link is in the support of any nontrivial homology representative, which implies $\mathrm{link}_{\Delta_i}(t_i) = \mathrm{link}_{\Delta}(t_i)$. ∎

THEOREM 6.3. *Coxeter complexes are saturated over any field of characteristic $p > 0$.*

Proof. Let Δ be a Coxeter complex with shelling $\sigma_1, \ldots, \sigma_m$. As above, let $\mathrm{res}(\sigma_i)$ be the restriction of σ_i in Δ_i and t_i be its complement. According to Corollary 4.2 it is enough to show that $\mathrm{link}_{\Delta_i}(t_i)$ is null for every i. But according to Lemma 6.2, $\mathrm{link}_{\Delta_i}(t_i) = \mathrm{link}_{\Delta}(t_i)$ is a Coxeter complex itself and so is null by Lemma 6.1. ∎

THEOREM 6.4. *Buildings are saturated over any field of characteristic $p > 0$.*

Proof. Let now Δ be a building with shelling $\sigma_1, \ldots, \sigma_m$. We shall keep the notation of the previous proof. Let $\Lambda_i := \mathrm{link}_{\Delta_i}(t_i)$. According to Theorem 4.1, it is enough to construct for every i a null-subcomplex of Λ_i coming through $\mathrm{res}(\sigma_i)$. We shall show that an apartment of $\mathrm{link}_{\Delta}(t_i)$ can be used for this.

For this note that in contrast to Coxeter complexes we know only that Λ_i is a subcomplex of $\mathrm{link}_{\Delta}(t_i)$ and that $\mathrm{link}_{\Delta}(t_i)$ has the homotopy type of a bouquet of $(k_i - 1)$-dimensional spheres. Furthermore, there is an apartment coming through $\mathrm{res}(\sigma_i)$ for each sphere in the bouquet and these apartments are Coxeter complexes, see Theorem 2 in [6, p. 93]. So the number of such apartments in Λ_i is the Euler characteristic of Λ_i, up to sign, or equivalently, the last component of the h-vector of Λ_i.

Now compare the complexes Λ_i and $\Lambda_i^* := \mathrm{link}_{\Delta_{i-1}}(t_i)$. Note that Λ_i is obtained from Λ_i^* by adding $\mathrm{res}(\sigma_i)$. This increases the last component of the h-vector by 1 and so this increases the number of apartments by 1. Consequently there is an apartment of $\mathrm{link}_{\Delta}(t_i)$ through $\mathrm{res}(\sigma_i)$ in Λ_i. ∎

To find the Betti numbers of a building for given prime p one may proceed as follows. It is well-known [5] that the h-vector of a building associated to a finite Chevalley group over $GF(q)$ with Weyl group (W, S) is given by

$$h_k(q) = \sum_{w \in (W, S):\, d(w) = k} q^{l(w)}$$

TABLE III

The h-Vector for the Building $A(6, q)$

$$
\begin{aligned}
h_0(q) &= 1;\\
h_1(q) &= 6q+10q^2+12q^3+16q^4+16q^5+18q^6+12q^7+12q^8+8q^9+\\
&\quad 6q^{10}+2q^{11}+2q^{12};\\
h_2(q) &= 10q^2+33q^3+58q^4+94q^5+120q^6+156q^7+159q^8+155q^9+\\
&\quad 135q^{10}+110q^{11}+74q^{12}+48q^{13}+27q^{14}+9q^{15}+3q^{16};\\
h_3(q) &= 4q^3+24q^4+56q^5+112q^6+164q^7+236q^8+292q^9+320q^{10}+\\
&\quad 320q^{11}+292q^{12}+236q^{13}+164q^{14}+112q^{15}+56q^{16}+24q^{17}+4q^{18};\\
h_4(q) &= 3q^5+9q^6+27q^7+48q^8+74q^9+110q^{10}+135q^{11}+155q^{12}+\\
&\quad 159q^{13}+156q^{14}+120q^{15}+94q^{16}+58q^{17}+33q^{18}+10q^{19};\\
h_5(q) &= 2q^9+2q^{10}+6q^{11}+8q^{12}+12q^{13}+12q^{14}+18q^{15}+16q^{16}+16q^{17}+\\
&\quad 12q^{18}+10q^{19}+6q^{20};\\
h_6(q) &= q^{21}
\end{aligned}
$$

where $l(w)$ is the length of $w \in W$ and $d(w) = |\{ s \in S : l(sw) < l(w) \}|$ is the number of descents of w. Now use Theorem 3.2 and the dimensions of $H^{n-j}_{(l-j,\, r-j)}$ given in [10] or [1]. To determine the isomorphism types of the homologies is a more difficult task. There is some progress on this question and these results will be the subject of a forthcoming paper.

We conclude with the example of the building $A(6, q)$ of dimension 5. Its h-vector is given in Table III. Its Betti numbers for $p = 3$ and $p = 5$ can be read off from the tables in Section 3.2, for other primes they are just as easy to construct.

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